



$f$  = fraction (0 to 1)

Q: At how many (0, 1, or 2) points does the spline curve cross the  $Y$  threshold, and what are the  $X$  coordinates of each?

Solution: Solve for  $f$ . If  $f$  is undefined, there are no threshold-crossing points. If  $f$  is defined, it will have two values — those values can be used to calculate their corresponding  $X$  values, but only when  $f$  is in the range 0 to 1 inclusive. If  $f$  is exactly 0 or 1, you must take into account whether the start/end point is *locally* going up or down from the  $Y$  threshold, just as you would for a line segment. (If it is locally horizontal, things get tricky — you must find out whether it curves off the horizontal *locally* in an up or down direction.)

$$Y = Sy + f(b - Sy) + f(b + f(Ey - b) - Sy - f(b - Sy))$$

$$Y = Sy + fb - fSy + f(b + fEy - fb - Sy - fb + fSy)$$

$$Y = Sy + fb - fSy + fb + f^2Ey - f^2b - fSy - f^2b + f^2Sy$$

$$0 = (Sy - Y) + f(2(b - Sy)) + f^2(Sy + Ey - 2b)$$

Then apply the quadratic formula to get:

$$f = \frac{(-2(b - Sy) [+ \text{ or } -] \sqrt{(2(b - Sy))^2 - 4(Sy + Ey - 2b)(Sy - Y)})}{(2(Sy + Ey - 2b))}$$

For each valid  $f$ , calculate its  $X$  coordinate as follows:

$$X = Sx + f(a - Sx) + f(a + f(Ex - a) - (Sx + f(a - Sx)))$$